

AN EXTENDED FOOTNOTE ON FINITELY MINIMAL MARTINGALE MEASURES

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ABSTRACT. This note contains a short discussion on the sufficiency of finite optimality in martingale transport. It is shown that finitely minimal martingale measures are solutions of the martingale transport problem when the cost function is upper semi-continuous and bounded from above by a sum of integrable functions. As an application a transparent proof of the uniqueness of left-monotone martingale transport plans is given.

Keywords: martingales, optimal transport, monotonicity principle.

1. INTRODUCTION AND RESULTS

Given probability measures μ and ν on \mathbb{R} , a transport or transport plan π is a probability measure on $\mathbb{R} \times \mathbb{R}$ that has μ and ν as its marginal measures. A martingale transport is any transport π that has the martingale property. This means that for each continuous and bounded function $\rho : \mathbb{R} \rightarrow \mathbb{R}$

$$\int \rho(x)(y - x) d\pi(x, y) = 0.$$

It is equivalent to require that for the disintegration $(\pi_x)_x$ of π one has

$$\int y d\pi_x(y) = x \text{ for } \mu\text{-a.e. } x.$$

The set of transport plans is denoted by $\mathcal{M}(\mu, \nu)$, the set of martingale transport plans by $\mathcal{M}_{\text{mart}}(\mu, \nu)$.

For a cost function $c : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$, the integral $\int c d\pi$ is the cost of the transport π . The Monge-Kantorovich problem with a martingale constraint that is considered in this note is to find

$$(\text{martMK}) \quad \inf_{\pi \in \mathcal{M}_{\text{mart}}(\mu, \nu)} \int c d\pi.$$

The set $\mathcal{M}_{\text{mart}}(\mu, \nu)$ is non-empty iff μ and ν are in convex order, i.e. $\mu \leq_{cx} \nu$. This relation of the marginals is hence always assumed.

Transport problems under martingale constraints and their connections to model-independent finance have been first studied in [HN12, BHL13, GH14]. Meanwhile, a rich literature on the subject exists. Without claiming completeness, we mention [ABPS16, BCH14, BJ16, BN14, BN15, CLM14, CKT15, DS14, GH14, HLTT14, HLT13, HK15, HN12, Jui14, Ste14, TT13, Zae15].

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As a particular contribution, [BJ16] started the investigation of finitely minimal martingale measures and the connection of finite minimality to optimality. It is in this context that this note should find its place. We repeat the relevant definitions:

Definition 1.1. Let α be a measure on $\mathbb{R} \times \mathbb{R}$. A measure α' is called a competitor of α if it has the same marginals as α and

$$\int y d\alpha_x(y) = \int y d\alpha'_x(y)$$

holds for $p_1(\alpha)$ -almost every x .

Definition 1.2. A set Γ is called c -finitely optimal if the following holds: whenever α is a finite measure supported on finitely many points in Γ , it is cost-minimizing amongst its competitors. I.e., if α' is a competitor of α , then

$$\int c d\alpha \leq \int c d\alpha'.$$

A martingale transport plan π is c -finitely optimal if there is a c -finitely optimal set Γ with $\pi(\Gamma) = 1$.

The idea behind this definition is to bring the concept of c -cyclical monotonicity to the martingale problem, and translate relevant results to the martingale case. Cyclical monotonicity of a transport plan in the usual Monge-Kantorovich problem is known to be equivalent to optimality under moderate conditions on the cost functions. For an overview and the details, readers are referred to [Vil09, AP03, ST09, BGMS09, Bei15]. It is already shown in [BJ16] that optimal martingale measures are c -finitely optimal. The reverse implication in [BJ16, Lemma 8.2] is less satisfactory: it assumes continuity and boundedness of the cost function. This note establishes a sufficiency result in more generality, covering also many types of cost functions not included before. For results of this type the name *monotonicity principle* has been suggested. See for instance Corollary 7.8 in [BNT16], where a result of similar spirit is proved.¹

Theorem 1.3. Let $\mu \leq_{cx} \nu$ be probability measures on \mathbb{R} . Let $c : \mathbb{R} \times \mathbb{R} \rightarrow [0, \infty)$ be bounded by a sum of integrable functions, i.e. let there be $c_1 \in L_1(\mu)$ and $c_2 \in L_1(\nu)$ such that

$$c(x, y) \leq c_1(x) + c_2(y)$$

holds for all $(x, y) \in A \times B$ where $\mu(A) = \nu(B) = 1$. Then any c -finitely optimal $\pi \in \mathcal{M}_{mart}(\mu, \nu)$ is optimal.

A particular contribution of [BJ16] was the introduction and investigation of *left-monotone* martingale transports:

Definition 1.4. A set $\Gamma \subseteq \mathbb{R} \times \mathbb{R}$ is called left-monotone if for all $(x, y^-), (x, y^+), (x', y') \in \Gamma$ with $x < x'$ and $y^- < y^+$ it follows that

$$y' \leq y^- \text{ or } y' \geq y^+.$$

A martingale transport π is called left-monotone if there is a left-monotone Γ with $\pi(\Gamma) = 1$.

¹The result in [BNT16] shows that if for a given cost function there are marginals μ and ν such that the dual problem admits a finite solution, then there is a (finitely minimal) set Λ such that any martingale measure γ (with arbitrary marginals) concentrated on Λ is optimal for the martingale transport problem associated to the marginals of γ .

[BJ16] shows existence, uniqueness and some remarkable optimality properties of a left-monotone martingale transport $\pi_{lc} \in \mathcal{M}_{\text{mart}}(\mu, \nu)$. However, the proof of uniqueness seems somewhat opaque as it relies on substantial preparations and a delicate approximation procedure. Thanks to the wider reach of Theorem 1.3, one can now almost immediately see from the construction of π_{lc} that it is the only left-monotone martingale transport. If one accepts Theorem 1.3 as an intuitive fact, this may be seen as a helpful abbreviation.

Theorem 1.5. *The martingale transport π_{lc} is the unique left-monotone element in $\mathcal{M}_{\text{mart}}(\mu, \nu)$.*

2. PROOF OF THEOREM 1.3

A successful strategy in deriving optimality from c -cyclical monotonicity in the usual Monge-Kantorovich problem was to show a splitting-property of c -cyclical monotone sets, see [ST09]: if Γ is c -cyclically monotone then there are measurable functions φ and ψ such that

$$\varphi(x) + \psi(y) \leq c(x, y)$$

holds everywhere and with equality on Γ . Although one does not need to know whether the functions φ and ψ are integrable (against μ and ν , respectively), one can still show that the integral of $\varphi \oplus \psi$ is invariant amongst the measures from $\mathcal{M}(\mu, \nu)$ for which this integral is defined. Optimality then follows easily.

As c -finite optimality is less than c -cyclical monotonicity this attempt cannot be copied verbatim, but it is possible to show a similar splitting property, namely that, after a suitable reduction of the problem, there are functions φ, ψ , and Δ such that

$$\varphi(x) + \psi(y) + \Delta(x)(y - x) \leq c(x, y).$$

holds everywhere and with equality on the c -finitely minimal set given. The strategy in this note is to examine the integrability problems that come with this splitting slightly more thoroughly and get the stronger result. This is done in the next definitions and lemmas. We assume that both μ and ν are concentrated on the interval I for which the statements are formulated. First we recall a simple result from [BJ16]:

Proposition 2.1. *Let $\chi : I \rightarrow \mathbb{R}$ be convex. Then for any $\pi \in \mathcal{M}_{\text{mart}}(\mu, \nu)$, the value*

$$J(\chi) = \int \int \chi(y) d\pi_x(y) - \chi(x) d\mu(x)$$

is well-defined in $[0, \infty]$ and does not depend on the choice of π .

Definition 2.2. Let $\varphi, \psi : I \rightarrow \mathbb{R}$ be measurable functions and $\chi : I \rightarrow \mathbb{R}$ a convex function such that both $\int \varphi - \chi d\mu < \infty$ and $\int \psi + \chi d\nu < \infty$. Then set

$$I(\varphi + \psi) = \int \varphi - \chi d\mu + \int \nu + \chi d\nu - J(\chi).$$

Note that this definition is, as the notation suggests, independent of the chosen function χ . Indeed, if χ and χ' are two functions fulfilling the conditions, then we can assume $I(\varphi + \psi, \chi)$ to be finite. Therefore $I(\varphi + \psi, \chi) - I(\varphi + \psi, \chi')$ is defined in $(-\infty, \infty]$. Computing the outcome yields 0, making use of the conditions on χ and χ' .

Lemma 2.3. *Let $\varphi, \psi, \Delta : I \rightarrow \mathbb{R}$ be measurable functions and $\pi \in \mathcal{M}_{\text{mart}}(\mu, \nu)$. If φ and ψ are as in definition 2.2 and the function*

$$\xi(x, y) = \varphi(x) + \psi(y) + \Delta(x)(y - x)$$

has a well-defined integral against π in $[-\infty, \infty)$, then

$$I(\varphi + \psi) = \int \int \varphi(x) + \psi(y) d\pi_x(y) d\mu(x) = \int \xi d\pi.$$

Proof.

$$\begin{aligned} I(\varphi + \psi) &= \int \varphi - \chi d\mu + \int \psi + \chi d\nu - J(\chi) \\ &= \int \varphi - \chi d\mu + \int \psi(y) + \chi(y) d\pi_x(y) d\mu(x) - J(\chi) \\ &= \int \left(\varphi(x) - \chi(x) + \int \psi(y) + \chi(y) d\pi_x(y) - \int \chi(y') d\pi_x(y') + \chi(x) \right) d\mu(x) \\ &= \int \left(\varphi(x) + \int \psi(y) d\pi_x(y) \right) d\mu(x) \\ &= \int \int \varphi(x) + \psi(y) d\pi_x(y) d\mu(x) \\ &= \int \xi d\pi. \end{aligned}$$

In the last step the martingale property of π was used. \square

Lemma 2.4. *Let $\varphi, \psi : I \rightarrow \mathbb{R}$ and $\Delta : I \rightarrow \mathbb{R}$ be functions, and let $c_1 \in L_1(\mu)$ and $c_2 \in L_1(\nu)$ be such that the inequality*

$$\varphi(x) + \psi(y) + \Delta(x)(y - x) \leq c_1(x) + c_2(y)$$

holds on a set $A \times B$ with $\mu(A) = \nu(B) = 1$. Then there is an interval $I' \subseteq I$ with $\mu(I') = \nu(I') = 1$, and a convex function $\chi : I' \rightarrow \mathbb{R}$ such that $\varphi - \chi \leq c_1$ μ -a.e. and $\psi + \chi \leq c_2$ ν -a.e.

Proof. On I define the functions

$$f = \varphi - c_1$$

and

$$g = \psi - c_2.$$

Define χ to be the largest convex function on I that is smaller than $-g$ on B , i.e.

$$\chi(x) = (-g)^{**}(x) = \sup \{ \alpha(x) : \alpha \text{ convex } I \rightarrow \mathbb{R} \text{ with } \alpha \leq -g \text{ on } B \}.$$

For any $x \in A$ the function $y \mapsto f(x) + \Delta(x)(y - x)$ is convex and not larger than $-g$ on B , hence $\chi \geq -\infty$. Also, $\chi < \infty$ on $I' = \text{conv } B$. We have $\nu(I') = 1$, and due to the convex order also $\mu(I') = 1$. For all $x \in A$ and all $y \in \text{conv } B$,

$$f(x) + \Delta(x)(y - x) \leq \chi(y).$$

Hence, setting $y = x$, for all $x \in A \cap I'$, and thus μ -a.e., $f(x) \leq \chi(x)$. Therefore,

$$\varphi - \chi \leq c_1 \quad \mu\text{-a.e.}$$

and

$$\psi + \chi \leq c_2 \quad \nu\text{-a.e.}$$

\square

Proof of Theorem 1.3. We first recall that it is enough to show the claim when the marginals are irreducible on an interval I : this means that $\mu(I) = \nu(I) = 1$, and that the potential function u_μ of μ is strictly smaller than the potential function u_ν of ν on I . That this does not lead to a loss of generality follows from Theorem 8.4 in [BJ16]: μ and ν can be decomposed into $\sum_{k \geq 0} \mu_k$, $\sum_{k \geq 0} \nu_k$ such that $\mu_0 = \nu_0$ and μ_k and ν_k are irreducible on I_k , where the intervals I_k are the components of $\{u_\mu < u_\nu\}$ and $\mu_k = \mu|_{I_k}$. Furthermore, any martingale transport between μ and ν can then be decomposed into a sum of martingale transports between μ_k and ν_k .

The irreducibility assumption can be used to show that a c -finitely optimal set Γ can be w.l.o.g. assumed to have additional regularity properties, and it can then be shown that there are measurable functions $\varphi, \psi : \mathbb{R} \rightarrow [-\infty, \infty)$ and $\Delta : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\varphi(x) + \psi(y) + \Delta(x)(y - x) \leq c(x, y)$$

holds everywhere and with equality on Γ . This is Proposition 8.10 in [BJ16].² Assume now that π is a c -finitely optimal transport plan, that is concentrated on a c -finitely optimal set Γ such that there are functions φ, ψ, Δ as above. After suitably restricting and changing the functions φ and ψ we are in the situation of Lemma 2.4, find a convex function χ as in Definition 2.2, and can define $I(\varphi + \psi)$. We set

$$\xi(x, y) = \varphi(x) + \psi(y) + \Delta(x)(y - x).$$

Due to the assumption on c being bounded by a sum of integrable functions, ξ is integrable for all transport plans. If $\pi' \in \mathcal{M}_{\text{mart}}(\mu, \nu)$, then due to Lemma 2.3

$$I(\varphi + \psi) = \int \xi d\pi$$

and

$$I(\varphi + \psi) = \int \xi d\pi'.$$

We also have $\int \xi d\pi = \int c d\pi$ and $\int \xi d\pi' \leq \int c d\pi'$, therefore

$$\int c d\pi \leq \int c d\pi'.$$

□

3. PROOF OF THEOREM 1.5

Before applying Theorem 1.3 to prove the claim, we recall how a particular (and as it turns out, *the*) left-monotone coupling π_{lc} is constructed. From this construction (or a characterization implied by it), Theorem 1.3 swiftly leads to uniqueness.

A measure μ is smaller in the extended convex order than a measure ν (notation: $\mu \leq_E \nu$) if for each nonnegative convex function φ

$$\int \varphi d\mu \leq \int \varphi d\nu.$$

For such measures there is a measure θ which is, (1), smaller than ν (i.e. $\theta \leq \nu$), but, (2), still larger than μ in the convex order (i.e. $\mu \leq_{cx} \theta$),

²The result in [BJ16] is only formulated for continuous cost functions, but it is not difficult to see that their proof also works for upper semi-continuous cost functions.

and, (3), smallest in convex order among all measures having the first two properties. θ is then called the shadow of μ in ν , notation: $S^\nu(\mu)$. Existence of the shadow is Lemma 4.6 in [BJ16]. The coupling π_{lc} is constructed via the following rule: let π_{lc} be the measure that transports, for each $s \in \mathbb{R}$, the restriction $\mu|_{(-\infty, s]}$ to its shadow in ν , i.e. to $S^\nu(\mu|_{(-\infty, s]})$. That this prescription leads indeed to a left-monotone measure in $\mathcal{M}_{\text{mart}}(\mu, \nu)$ is Theorem 4.18 in [BJ16]. Most important for us is an easy consequence of this: letting $c_{s,t}$ denote the cost function $(x, y) \mapsto \mathbb{1}_{(-\infty, s]}(x)|y - t|$, the measure π_{lc} is the unique measure in $\mathcal{M}_{\text{mart}}(\mu, \nu)$ that is a minimizer for all the functions $c_{s,t}$. This follows easily from the definition of the shadow and the fact that the functions $y \mapsto |y - t|$ are enough to describe the convex order of measures on \mathbb{R} .

Lemma 3.1. *A left-monotone set Γ is $c_{s,t}$ -finitely minimal for all $s, t \in \mathbb{R}$.*

Proof. Let α be a finite measure on finitely many points in Γ , let α' be a competitor of α and let $x_1 < \dots < x_n$ be the points on which $p_1(\alpha) = p_1(\alpha')$ is concentrated. If $s < x_1$ or $s \geq x_n$, then $\int c_{s,t} d\alpha = \int c_{s,t} d\alpha'$. In slight misuse of notation, writing α_{x_i} for the one-dimensional measure $\alpha|_{\{x_i\} \times \mathbb{R}}$ (i.e. the non-normalized disintegration of α), we claim that

$$\alpha_{x_1} + \dots + \alpha_{x_i} \leq_{cx} \alpha'_{x_1} + \dots + \alpha'_{x_i}$$

holds for all $i = 1, \dots, n$. This in particular implies that $\int c_{s,t} d\alpha \leq \int c_{s,t} d\alpha'$ also holds for all $s \in [x_1, x_n]$:

Let $i = 1$: set $y_1^- = \min\{y : \alpha(x_1, y) > 0\}$ and $y_1^+ = \max\{y : \alpha(x_1, y) > 0\}$. By left-monotonicity, for all $y \in (y_1^-, y_1^+)$ we must have

$$\alpha(x_1, y) = p_1(\alpha)(y) \geq \alpha'(x_1, y).$$

But $\int y d\alpha_{x_1}(y) = \int y d\alpha'_{x_1}(y)$ and $\sum_y \alpha_{x_1}(y) = \sum_y \alpha'_{x_1}(y)$, hence

$$\alpha_{x_1} \leq_{cx} \alpha'_{x_1}.$$

Let $i = 2$: If $y_2^- = \min\{y : \alpha(x_2, y) > 0\} \geq y_1^+$ or if $y_2^+ = \max\{y : \alpha(x_2, y) > 0\} \leq y_1^-$, then one can repeat the above argument for $\alpha_{x_2}, \alpha'_{x_2}$ and the claim follows by addition. Elsewise, one can repeat the argument for $\alpha_{x_1} + \alpha_{x_2}, \alpha'_{x_1} + \alpha'_{x_2}$ in place of $\alpha_{x_1}, \alpha'_{x_1}$.

For larger i only the notation is more complicated. \square

Proof of Theorem 1.5. Let $\pi \in \mathcal{M}_{\text{mart}}(\mu, \nu)$ be left-monotone and Γ be a left-monotone set with $\pi(\Gamma) = 1$. By the above lemma, Γ is $c_{s,t}$ -finitely minimal for all $s, t \in \mathbb{R}$. By Theorem 1.3 it is hence a minimizer for all $c_{s,t}$, and it must therefore be equal to π_{lc} . \square

REFERENCES

- [ABPS16] B. Acciaio, M. Beiglöck, F. Penkner, and W. Schachermayer. A model-free version of the fundamental theorem of asset pricing and the super-replication theorem. *Math. Finance*, 26(2):233–251, 2016.
- [AP03] L. Ambrosio and A. Pratelli. Existence and stability results in the L^1 theory of optimal transportation. In *Optimal transportation and applications (Martina Franca, 2001)*, volume 1813 of *Lecture Notes in Math.*, pages 123–160. Springer, Berlin, 2003.
- [BCH14] M. Beiglöck, A. Cox, and M. Huesmann. Optimal transport and skorokhod embedding. *arxiv:1307.3656v1*, 2014.
- [Bei15] M. Beiglöck. Cyclical monotonicity and the ergodic theorem. *Ergodic Theory Dynam. Systems*, 35(3):710–713, 2015.

- [BGMS09] M. Beiglböck, M. Goldstern, G. Maresch, and W. Schachermayer. Optimal and better transport plans. *J. Funct. Anal.*, 256(6):1907–1927, 2009.
- [BHL13] M. Beiglböck, P. Henry-Labordère, and F. Penkner. Model-independent bounds for option prices: A mass transport approach. *Finance Stoch.*, 17(3):477–501, 2013.
- [BJ16] M. Beiglböck and N. Juillet. On a problem of optimal transport under marginal martingale constraints. *Ann. Probab.*, 44(1):42–106, 2016.
- [BN14] M. Beiglböck and M. Nutz. Martingale inequalities and deterministic counterparts. *Electron. J. Probab.*, 19(95):1–15, 2014.
- [BN15] B. Bouchard and M. Nutz. Arbitrage and duality in nondominated discrete-time models. *Ann. Appl. Probab.*, 25(2):823–859, 2015.
- [BNT16] M. Beiglböck, M. Nutz, and N. Touzi. Complete duality for martingale optimal transport on the line. *Ann. Probab.*, to appear, 2016.
- [CKT15] P. Cheridito, M. Kupper, and L. Tangpi. Representation of increasing convex functionals with countably additive measures. *arxiv:1502.05763v1*, 2015.
- [CLM14] L. Campi, I. Laachir, and C. Martini. Change of numeraire in the two-marginals martingale transport problem. *arxiv:1406.6951v3*, 2014.
- [DS14] J. Dolinsky and H. M. Soner. Martingale optimal transport and robust hedging in continuous time. *Probab. Theory Relat. Fields*, 160(1-2):391–427, 2014.
- [GHLT14] A. Galichon, P. Henry-Labordère, and N. Touzi. A stochastic control approach to no-arbitrage bounds given marginals, with an application to lookback options. *Ann. Appl. Probab.*, 24(1):312–336, 2014.
- [HK15] D. Hobson and M. Klimmek. Robust price bounds for the forward starting straddle. *Finance Stoch.*, 19(1):189–214, 2015.
- [HLT13] P. Henry-Labordère and N. Touzi. An explicit martingale version of brenier’s theorem. *To appear in Finance Stoch.*, 2013.
- [HLTT14] P. Henry-Labordère, X. Tan, and N. Touzi. An explicit version of the one-dimensional brenier’s theorem with full marginal constraints. *Stochastic Process. Appl.*, to appear, 2014.
- [HN12] D. Hobson and A. Neuberger. Robust bounds for forward start options. *Mathematical Finance*, 22(1):31–56, December 2012.
- [Jui14] N. Juillet. Stability of the shadow projection and the left-curtain coupling. *Ann. Inst. H. Poincaré Probab. Stat.*, to appear, 2014.
- [ST09] Walter Schachermayer and Josef Teichmann. Characterization of optimal transport plans for the Monge-Kantorovich problem. *Proc. Amer. Math. Soc.*, 137(2):519–529, 2009.
- [Ste14] F. Stebegg. Model-independent pricing of asian options via optimal martingale transport. *arxiv:1412.1429v1*, 2014.
- [TT13] X. Tan and N. Touzi. Optimal transportation under controlled stochastic dynamics. *Ann. Probab.*, 41(5):3201–3240, 2013.
- [Vil09] C. Villani. *Optimal transport - Old and New*. Grundlehren der mathematischen Wissenschaften. Springer, 2009.
- [Zae15] D. Zae. On the monge-kantorovich problem with additional linear constraints. *Math. Notes*, 98(5):725–741, 2015.